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Subdominant critical indices for the ferromagnetic susceptibility of the spin-$\frac{1}{2}$ Ising model

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Abstract. Starting from the high-temperature series for the susceptibility of the spin-$\frac{1}{2}$ ferromagnetic Ising models — square, planar triangular, SC, BCC, FCC, diamond, and HSC in four, five and six dimensions—we analyse the analytical structure near the critical point of the susceptibility by writing it as a Laplace transform in the variable $\log(1-w/w_c)$, where $w=\tanh(\beta J)$.

The interest of the method is that the coalescing singularities which sit at $w=w_c$ are spread out and can be analysed separately, the first subdominant critical indices appearing as stable poles of Padé approximants. We first recover the results for the two-dimensional models, with a high accuracy: $\gamma=1\cdot74995$ and $\gamma_s=0\cdot757$. The most stable results in three dimensions are obtained for the diamond lattice: $\gamma=1\cdot2506\pm0\cdot0015$ and $\gamma_s=0\cdot42\pm0\cdot11$. The other lattices give $0\cdot10<\gamma_s<0\cdot52$. However, in all three-dimensional cases the amplitude of the subdominant singularity is less than a few per cent of the leading one. Therefore the uncertainties are large, but the subdominant singularity stays closer to the value $\gamma-1=0\cdot25$ than to the field theoretical predicted one $\gamma_s=0\cdot75$.

1. Introduction

When calculating the susceptibility critical index γ from the high-temperature series expansion, by using Padé approximations on the logarithmic derivative series (for a review see Gaunt and Guttmann (1974)), one is always aware of the fact that the presence of other singularities superimposed on the leading one can limit the precision of the calculation. It is therefore interesting to use a method of analysis which can in principle disentangle the various singularities, in such a way as to leave the leading singularities as free as possible from the influence of the other superimposed ones.

Baker and Hunter (1973) proposed a method for achieving such a program. In this paper we shall systematically use the Baker–Hunter method, which we present in § 2, from a slightly different and complementary point of view.

Baker and Hunter analysed the singularities of the ferromagnetic susceptibility of the $s=\frac{1}{2}$ Ising model for various lattices. They found no evidence for a subdominant singularity. However, a significantly larger number of terms of the series is now available (six more in the diamond lattice case). It is therefore interesting to re-examine the behaviour of the dominant index γ and look for possible subdominant singularities.

Similar analyses have been performed for various values of the spin (Camp and Van Dyke 1975, Camp *et al* 1976, Moore *et al* 1979). Due to the difficulty of handling the general spin case, only 12 terms in the susceptibility series of the FCC lattice were

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considered. The first subdominant singularity $\gamma_s \sim 0\cdot75 \pm 0\cdot08$ was found compatible with the field theoretical prediction. However, the residue of this subdominant singularity vanishes in the spin-$\frac{1}{2}$ case (Saul *et al* 1975).

In addition, the values of γ obtained from the susceptibility series were quoted to be (Camp and Van Dyke 1975) $\gamma = 1\cdot250\ (^{+0\cdot003}_{-0\cdot007})$. They remain slightly (Gaunt and Sykes 1973) but significantly larger than the most recent values obtained from field theoretical methods (Baker *et al* 1976, Le Guillou and Zinn-Justin 1977). These authors computed the critical indices in the n-vector model using a method which incorporates asymptotic estimates of large order of the perturbative series. They obtain $\gamma = 1\cdot2402 \pm 0\cdot0009$. The classical '$D$ log Padé' analysis (Gaunt and Guttmann 1974) gives $\gamma = 1\cdot250 \pm 0\cdot001$ for the three-dimensional Ising model, using 17 terms of the susceptibility series.

To try to clarify this situation we were led to analyse the susceptibility series (Domb 1974, Sykes *et al* 1972, Gaunt and Sykes 1973) for the follwing lattices: planar triangular (15 terms), square (21 terms), SC (Gaunt and Sykes 1979) (19 terms), diamond (21 terms), BCC (15 terms), FCC (McKenzie 1975) (15 terms). Results for the HSC lattices (Fischer and Gaunt 1964, Baker 1977) are also given in four, five and six dimensions, using only 12-term expansions.

Of course we have deliberately restricted our analysis to the susceptibility series, since other thermodynamical quantities are known only with a significantly smaller number of terms. We were particularly interested in the variations of the susceptibility dominant and subdominant indices with the order of approximation.

As explained in what follows, our analysis is by no means a fit, but a search for what could be reasonably assumed on the analytic structure of the susceptibility series. In addition, our intention was mainly to compare results obtained in various dimensions using the same set of Padé approximants. Our analysis uses only the subdiagonal $[N-1/N]$ Padé approximants, which give precise results in two dimensions.

2. The method

We consider the susceptibility $\chi(w)$, in zero magnetic field, where

$$w = \tanh(\beta J). \tag{2.1}$$

We shall introduce also the critical value

$$w_c = \tanh(\beta_c J) \tag{2.2}$$

corresponding to the critical temperature β_c.

It is convenient to write $\chi(w)$ in the form of a Laplace transform:

$$\chi(w) = \int_\gamma^\infty \left(1 - \frac{w}{w_c}\right)^\lambda \sigma(\lambda)\, d\lambda, \qquad 0 < w < w_c \tag{2.3}$$

Let us first analyse the possible structure of $\sigma(\lambda)$.

(i) If $\sigma(\lambda)$ is a pure point spectrum of the form

$$\sigma(\lambda) = \sum_{n=0}^{\infty} \sigma_n \delta(\lambda - \gamma_n), \tag{2.4}$$

we see that

$$\chi(w) = \sum_{n=0}^{\infty} \sigma_n \left(1 - \frac{w}{w_c}\right)^{\gamma_n}. \tag{2.5}$$

Among the γ_n we can include the set of natural numbers 0, 1, 2, ..., and therefore the analytic structure of $\chi(w)$ near w_c is rather simple: it is made of an infinite (or possibly finite) set of coalescing logarithm-type singularities, plus a regular part coming from the set of integer values of the γ_n.

A slightly more refined version of the point spectrum is to allow also for δ function derivatives, and so we shall consider

$$\sigma(\lambda) = \sum_{n,p=0}^{\infty} \sigma_{n,p} (-)^p \delta^{(p)}(\lambda - \gamma_n), \tag{2.6}$$

which gives rise to

$$\chi(w) = \sum_{n,p=0}^{\infty} \sigma_{n,p} \left(1 - \frac{w}{w_c}\right)^{\gamma_n} \log^p\left(1 - \frac{w}{w_c}\right). \tag{2.7}$$

(ii) If $\sigma(\lambda)$ has a continuous part in it, we can obtain a very rich structure; in particular, terms of the form

$$\left(1 - \frac{w}{w_c}\right)^{\gamma_n} \left[-\log\left(1 - \frac{w}{w_c}\right)\right]^{-\alpha_n}, \tag{2.8}$$

where α_n is not a natural number. This is provided by a measure of the form

$$\frac{(\lambda - \gamma_n)^{\alpha_n - 1}}{\Gamma(\alpha_n)} \theta(\lambda - \gamma_n). \tag{2.9}$$

Therefore an analysis of the measure $\sigma(\lambda)$ gives a deep insight into the analytic structure of $\chi(w)$.

Associated with the measure $\sigma(\lambda)$ we introduce its Hilbert transform

$$S(w) = \int_{\gamma}^{\infty} \frac{\sigma(\lambda)\, \mathrm{d}\lambda}{1 - \lambda w}. \tag{2.10}$$

We see that $S(w)$ will be a meromorphic function of w of the form

$$S(w) = \sum_{n=0}^{\infty} \frac{\sigma_n}{1 - w\gamma_n} \tag{2.11}$$

when $\chi(w)$ is of the simple form (2.5).

If $\chi(w)$ is of the form (2.7), then

$$S(w) = \sum_{n,p=0}^{\infty} \sigma_{n,p} \frac{\gamma_n^p p!}{(1 - w\gamma_n)^{p+1}}. \tag{2.12}$$

By analysing $S(w)$ in terms of Padé approximants we shall immediately recognise the nature of the singularities of $S(w)$: if a pole of the Padé approximant is isolated and stable with the change of order of approximation, we shall have a contribution to $\chi(w)$ of the simplest form. Double poles or doublets of poles very close to each other will indicate the presence of a logarithm in $\chi(w)$. Real poles very unstable, by changing the order of approximation, will show the existence of a continuous spectrum for the measure.

Therefore the analysis of $S(w)$ in terms of Padé approximants appears to provide much more than a simple fit for $\chi(w)$: it may show what could be the analytical structure of $\chi(w)$ near w_c.

It is possible to derive from the Taylor series expansion of $\chi(w)$ near $w=0$ the Taylor series expansion of $S(w)$ around $w=0$, at the same order, by a finite linear transformation. We write

$$\chi(w)=\int_\gamma^\infty\left(1-\frac{w}{w_c}\right)^\lambda\sigma(\lambda)\,\mathrm{d}\lambda=\sum_{k=0}^\infty(-)^k\frac{w^k}{w_c^k}\int_\gamma^\infty\binom{\lambda}{k}\sigma(\lambda)\,\mathrm{d}\lambda=\sum_{k=0}^\infty\rho_k w^k,$$

from which we obtain

$$\mu_{[k]}=(-)^k w_c^k k!\rho_k, \tag{2.13}$$

where $\mu_{[k]}$ is the factorial moment of the measure $\sigma(\lambda)$:

$$\mu_{[k]}=\int_\gamma^\infty\lambda(\lambda-1)\ldots(\lambda-k+1)\sigma(\lambda)\,\mathrm{d}\lambda. \tag{2.14}$$

From the factorial moments we obtain the moments

$$\mu_k=\int_\gamma^\infty\lambda^k\sigma(\lambda)\,\mathrm{d}\lambda, \tag{2.15}$$

which are precisely the Taylor series coefficients of $S(w)$. We can then construct the Padé approximants for $S(w)$. Numerical calculations are made easier by using the recursion formulae

$$\mu_k=\mu_{[k]}-\sum_{j=0}^{k-1}C_{k,j}\mu_j,\qquad k=1,2,\ldots, \tag{2.16}$$

where the $C_{k,j}$ are obtained from the relations

$$C_{n+1,j}=C_{n,j-1}-nC_{n,j},\qquad j=1,2,\ldots n,\quad n=1,2,\ldots, \tag{2.17}$$

with the auxiliary conditions

$$C_{n,n}=1,\qquad C_{n,0}=0. \tag{2.18}$$

The interested reader will find in the work of Baker and Hunter (1973) numerical examples on explicit test functions to which the previous method has been applied.

To end this section it is interesting to remark that for $\chi(w)$ to admit the representation (2.3) it is necessary to have a minimal domain of analyticity in w. This domain is made of the interior of the circle of centre w_c passing through the origin and cut from w_c to $2w_c$.

3. Discussions of the results

We shall first consider the results obtained in two dimensions by analysing the square lattice and the planar triangular lattice. This analysis is important, because exact results are known for these cases, and we shall see how our approximations reproduce them.

3.1. *The two-dimensional models*

We write

$$\chi(w)=\chi_0\left(1-\frac{w}{w_c}\right)^{-\gamma_0}+\chi_1\left(1-\frac{w}{w_c}\right)^{-\gamma_1}+\dots \tag{3.1}$$

From now on, to follow standard notations, we shall use γ instead of γ_0, and γ_s instead of γ_1, the corresponding amplitudes being χ and χ_s. For the square lattice the exact results are (Wu *et al* 1976)

$$\begin{aligned}\gamma&=\tfrac{7}{4}, & \chi&=0\cdot7717340,\\ \gamma_s&=\tfrac{3}{4}, & \chi_s&=0\cdot34790.\end{aligned} \tag{3.2}$$

Table 1 gives the values of the successive subdominant indices, as well as their associated amplitudes χ_k, produced by the subdiagonal Padé $[N-1/N]$ approximants of increasing order. In these calculations we have used the exact value of $w_c=\sqrt{2}-1$; the effect of an error in w_c, which is unavoidable in three dimensions, will be discussed later. We see in table 1 that, while the first 12 terms of the expansion already yield an excellent value for γ, at least 18 terms are necessary to reach stability for γ_s.

Table 1. The square lattice model. Calculation of the subcritical indices $\chi(w)=\Sigma_{k=0}^{\infty}\chi_k(1-w/w_c)^{-\gamma_k}$; $w_c=\sqrt{2}-1$ (exact value).

Number of of terms	γ_0	χ_0	γ_1	χ_1	γ_2	χ_2
12	1·7469	0·7818	0·6282	0·432	0·024	−0·207
14	1·7437	0·7909	0·4572	0·939	0·245	−0·724
16	1·7407	0·7989	0·369-0·2i	0·10+0·4i	0·369+0·2i	0·10−0·4i
18	1·7509	0·7686	0·7896	0·3225	−0·545	−0·10
20	1·74998	0·7717	0·75797	0·3359	−0·324	−0·10
22	1·74995	0·7719	0·7567	0·3365	−0·317	−0·10
∞	1·750000	0·771734	0·750000	0·34790		

At order 18 we have

$$\begin{aligned}\gamma^{(18)}&=1\cdot7509, & \chi^{(18)}&=0\cdot7686,\\ \gamma_s^{(18)}&=0\cdot7896, & \chi_s^{(18)}&=0\cdot3225.\end{aligned} \tag{3.3}$$

At order 22, we have

$$\begin{aligned}\gamma^{(22)}&=1\cdot74995, & \chi^{(22)}&=0\cdot7719,\\ \gamma_s^{(22)}&=0\cdot7567, & \chi_s^{(22)}&=0\cdot3365,\end{aligned} \tag{3.4}$$

to be compared with the exact values given in (3.2). We remark in table 1 that all poles of the Padé approximants are well separated and very stable with respect to a change of order. Even the second subdominant γ_2 value is not far from the expected value of $-\frac{1}{4}$. Furthermore, the amplitudes of γ_3, γ_4 and γ_5 are extremely small, and these poles are far away: a formula with three poles represents $\chi(w)$ extremely well. This is in agreement with a similar statement made by Sykes *et al* (1972).

More important is to see how an error in w_c of the order of a few times 10^{-5} affects γ and γ_s. We find that the variations $\Delta\gamma$ and $\Delta\gamma_s$ vary linearly with Δw_c for $\Delta w_c/w_c$ ranging from -9×10^{-5} to $+9\times10^{5}$:

$$\begin{aligned} \frac{\Delta\gamma}{\gamma} &= a_\gamma\frac{\Delta w_c}{w_c}, \qquad & a_\gamma &\leqslant 33, \\ \frac{\Delta\gamma_s}{\gamma_s} &= a_s\frac{\Delta w_c}{w_c}, \qquad & a_s &\leqslant 1500. \end{aligned} \tag{3.5}$$

For a typical error in w_c of the order of 5×10^{-5}, which would correspond to the precision achieved for the three-dimensional models (Domb 1974), we shall have errors of the order of $1\cdot5\times10^{-3}$ for γ and 7% for γ_s. These errors are much larger than the errors coming from the series truncation: therefore we shall use them to define the precision achieved on γ and γ_s.

Analysis of the planar triangular model provides essentially the same conclusion as for the square model. However, because only 16 terms of the susceptibility expansion are available, the results are slightly less accurate: see table 2. The exact values quoted in table 2 are obtained from Guttmann (1974, 1977).

Table 2. The planar triangular model. Calculation of the subcritical indices $\chi(w)=\sum_{k=0}^{\infty}\chi_k(1-w/w_c)^{-\gamma_k}$; $w_c=2-\sqrt{3}$ (exact value).

Number of terms	γ_0	χ_0	γ_1	χ_1	γ_2	χ_2
12	1·7518	0·841	0·846	0·16	$-1\cdot1-2i$	$(-4-3i)\times10^{-4}$
14	1·7520	0·840	0·850	0·161	$-1\cdot1-2i$	$(-4-3i)\times10^{-4}$
16	1·7503	0·846	0·789	0·162	$-1-0\cdot4i$	$(-4-10i)\times10^{-3}$
∞	1·750000	0·8471	0·75000	0·1758		

At order 16 we have

$$\begin{aligned} \gamma^{(16)} &= 1\cdot7503, \qquad & \chi^{(16)} &= 0\cdot846, \\ \gamma_s^{(16)} &= 0\cdot789, \qquad & \chi_s^{(16)} &= 0\cdot162. \end{aligned} \tag{3.6}$$

We also see that poles of the approximation are stable and well separated from each other, and that their effect becomes negligible after the third.

3.2. *The three-dimensional models*

It is now with confidence that we shall apply the previous analysis to the three-dimensional models.

The Taylor series expansions for the susceptibilities, as well as the best critical temperature values, have been taken from Domb (1974). We shall first analyse the diamond model in detail for which 22 terms of the expansion are available.

Table 3 gives the numerical results of the analysis. We remark, as previously, that, while only 12 terms are necessary to reach an already stable and precise value of γ, it is

Table 3. The diamond model. Calculation of the subcritical indices $\chi(w) = \Sigma_{k=0}^{\infty} \chi_k (1 - w/w_c)^{-\gamma_k}$; $w_c^{-1} = 2{\cdot}82641$.

Number of terms	γ_0	χ_0	γ_1	χ_1	γ_2	χ_2
12	1·2439	1·066	$-1{\cdot}3-0{\cdot}9\mathrm{i}$	$-(3+0{\cdot}3\mathrm{i})\times 10^{-2}$	$-1{\cdot}3+0{\cdot}9\mathrm{i}$	$-(3-0{\cdot}3\mathrm{i})\times 10^{-2}$
14	1·2436	1·067	$-1{\cdot}4-0{\cdot}9\mathrm{i}$	$(-3-0{\cdot}2\mathrm{i})\times 10^{-2}$	$-1{\cdot}4+0{\cdot}9\mathrm{i}$	$-(3+0{\cdot}2\mathrm{i})\times 10^{-2}$
16	1·2509	1·043	0·457	$4{\cdot}8\times 10^{-2}$	$-1{\cdot}1$	-9×10^{-2}
18	1·2503	1·046	0·385	$4{\cdot}8\times 10^{-2}$	$-1{\cdot}0$	-9×10^{-2}
20	1·2509	1·043	0·468	$4{\cdot}8\times 10^{-2}$	$-1{\cdot}1$	-9×10^{-2}
22	1·2506	1·045	0·424	$4{\cdot}8\times 10^{-2}$	$-1{\cdot}1$	-9×10^{-2}

necessary to go beyond 16 terms to stabilise γ_s. With 22 terms we get

$$\begin{aligned} \gamma^{(22)} &= 1{\cdot}2506, & \chi^{(22)} &= 1{\cdot}045, \\ \gamma_s^{(22)} &= 0{\cdot}424, & \chi_s^{(22)} &= 4.7\times 10^{-2}, \end{aligned} \tag{3.7}$$

values achieved with $w_c^{-1} = 2{\cdot}82641$.

We notice that the value of the first subdominant amplitude is small compared with the two-dimensional case. However, all values of all poles and residues are stable when going from 16 to 22 terms of the perturbation series. We consider now the effect on γ and γ_s of an error in w_c. Taking the value $\Delta w_c/w_c \simeq 4\times 10^{-5}$ we have reproduced in table 4 the changes we get on the critical and subdominant indices.

Table 4. Effect on the critical and subcritical indices of the three-dimensional models due to an error in the critical temperature w_c.

Model	$\Delta w_c/w_c$	γ_0	χ_0	γ_1	χ_1
SC	4×10^{-5}	$1{\cdot}2493^{-0{\cdot}0008}_{+0{\cdot}0012}$	$1{\cdot}019^{+0{\cdot}002}_{-0{\cdot}005}$	$0{\cdot}36^{-0{\cdot}30}_{+0{\cdot}30}$	$\left(8{\cdot}5^{-0{\cdot}7}_{+5}\right)\times 10^{-3}$
BCC	$1{\cdot}6\times 10^{-4}$	$1{\cdot}2478^{-0{\cdot}004}_{+0{\cdot}0042}$	$0{\cdot}974^{+0{\cdot}011}_{-0{\cdot}013}$	$0{\cdot}38^{-0{\cdot}15}_{+0{\cdot}19}$	$(3{\cdot}7^{-2{\cdot}7}_{+1})\times 10^{-2}$
FCC	5×10^{-5}	$1{\cdot}2458^{-0{\cdot}0015}_{+0{\cdot}0005}$	$0{\cdot}977^{+0{\cdot}004}_{-0{\cdot}019}$	$0{\cdot}13^{\text{(complex)}}_{+0{\cdot}10}$	$(3{\cdot}5^{\text{complex}}_{-0{\cdot}8})\times 10^{-2}$
Diamond	4×10^{-5}	$1{\cdot}2506^{-0{\cdot}014}_{+0{\cdot}0015}$	$1{\cdot}0447^{+0{\cdot}0046}_{-0{\cdot}0057}$	$0{\cdot}42^{-0{\cdot}11}_{+0{\cdot}10}$	$(4{\cdot}7^{-0{\cdot}1}_{+0{\cdot}3})\times 10^{-2}$

In the same way we have analysed the SC model (see table 5). Here we have less terms of the expansion available, and we obtain

$$\begin{aligned} \gamma^{(20)} &= 1{\cdot}2493, & \chi^{(20)} &= 1{\cdot}019, \\ \gamma_s^{(20)} &= 0{\cdot}364, & \chi_s^{(20)} &= 8{\cdot}5\times 10^{-3}. \end{aligned} \tag{3.8}$$

Again all results are comparable with the case of the diamond.

For the BCC (table 6) model we get

$$\begin{aligned} \gamma^{(16)} &= 1{\cdot}2478, & \chi^{(16)} &= 0{\cdot}974, \\ \gamma_s^{(16)} &= 0{\cdot}385, & \chi_s^{(16)} &= 3{\cdot}6\times 10^{-2}, \end{aligned} \tag{3.9}$$

Table 5. The SC model. Calculation of the subcritical indices $\chi(w)=\Sigma_{k=0}^{\infty}\chi_k(1-w/w_c)^{-\gamma_k}$; $w_c^{-1}=4{\cdot}5844$.

Number of terms	γ_0	χ_0	γ_1	χ_1	γ_2	χ_2
12	1·2494	1·019	0·3979	$8{\cdot}6\times10^{-3}$	−1·16	$-2{\cdot}7\times10^{-2}$
14	1·2493	1·019	0·3551	$8{\cdot}4\times10^{-3}$	−1·15	$-2{\cdot}7\times10^{-2}$
16	1·2495	1·018	0·4620	$8{\cdot}6\times10^{-3}$	−1·2	$-2{\cdot}7\times10^{-2}$
18	1·2494	1·019	0·391	$8{\cdot}5\times10^{-3}$	−1·2	$-2{\cdot}7\times10^{-2}$
20	1·2493	1·019	0·3636	$8{\cdot}5\times10^{-3}$	−1·15	$-2{\cdot}7\times10^{-2}$

Table 6. The BCC model. Calculation of the subcritical indices $\chi(w)=\Sigma_{k=0}^{\infty}\chi_k(1-w/w_c)^{-\gamma_k}$; $w_c^{-1}=6{\cdot}4055$.

Number of terms	γ_0	χ_0	γ_1	χ_1	γ_2	χ_2
12	1·2491	0·969	0·494	4×10^{-2}	−1·86	$-8{\cdot}9\times10^{-3}$
14	1·2481	0·973	0·412	$3{\cdot}7\times10^{-2}$	−1·68	$-9{\cdot}6\times10^{-3}$
16	1·2478	0·974	0·385	$3{\cdot}7\times10^{-2}$	−1·59	$-9{\cdot}9\times10^{-3}$

and for the FCC (table 7) model we have

$$\gamma^{(16)}=1{\cdot}2458,\qquad \chi^{(16)}=0{\cdot}977,$$
$$\gamma_s^{(16)}=0{\cdot}13,\qquad \chi_s^{(16)}=3{\cdot}5\times10^{-2}. \tag{3.10}$$

Table 7. The FCC model. Calculations of the subcritical indices $\chi(w)=\Sigma_{k=0}^{\infty}\chi_k(1-w/w_c)^{-\gamma_k}$; $w_c^{-1}=9{\cdot}830$.

Number of terms	γ_0	χ_0	γ_1	χ_1	γ_2	χ_2
8	1·2507	0·959	−0·572	0·038	0·482 + 2i	mod < 10^{-3}
10	1·2458	0·976	0·053	0·029	0·903	−0·058
12	1·2457	0·977	0·126	0·034	0·677	−0·011
14	1·2457	0·977	0·176	0·041	0·559	−0·189
16	1·2458	0·977	0·135	0·035	0·652	−0·12

With the exception of the FCC model, for which the Padé approximants exhibit anomalous complex poles and fast variation with the critical value w_c, table 4 shows that all the other models give consistent results. The most stable values are obtained for the diamond, for which a large number of terms is available. In addition the variation of γ_s with w_c is slow. For the diamond we have

$$\gamma_s=0{\cdot}42^{-0{\cdot}11}_{+0{\cdot}10},\qquad \gamma=1{\cdot}2506^{-1{\cdot}4\times10^{-3}}_{1{\cdot}5\times10^{-3}}.$$

If we want to include all three-dimensional models, we find that the inequality $0{\cdot}08<\gamma_s<0{\cdot}52$ is compatible with all models.

3.3. The HSC models in dimensions four, five and six

We give, for completeness, the results for models of dimension greater than three. For all these models only 12 expansion terms are available.

Tables 8–10 show the great stability of the results; the dominant singularity is compatible with the classical value $\gamma = 1$.

Table 8. The HSC model in four dimensions. Calculation of the subcritical indices $\chi(w) = \Sigma_{k=0}^{\infty} \chi_k (1 - w/w_c)^{-\gamma_k}$; $w_c = 0{\cdot}148766$.

Number of terms	γ_0	χ_0	γ_1	χ_1	γ_2	χ_2
8	1·1023	1·077	0·057	$-7{\cdot}4\times10^{-2}$	−2.5	-3×10^{-3}
10	1·0956	1·100	0·305	$-9{\cdot}2\times10^{-2}$	−1·5	-8×10^{-3}
12	1·0926	1·114	0·410	$-1{\cdot}0\times10^{-1}$	−1·3	-1×10^{-2}

Table 9. The HSC model in five dimensions. Calculation of the subcritical indices $\chi(w) = \Sigma_{k=0}^{\infty} \chi_k (1 - w/w_c)^{-\gamma_k}$; $w_c = 0{\cdot}113541$.

Number of terms	γ_0	χ_0	γ_1	χ_1	γ_2	χ_2
8	1·0380	1·1217	0·289	−0·119	−2·36	$-2{\cdot}3\times10^{-3}$
10	1·0335	1·1410	0·372	−0·137	−1·85	$-3{\cdot}8\times10^{-3}$
12	1·0331	1·1430	0·381	−0·139	−1·79	$-4{\cdot}0\times10^{-3}$

Table 10. The HSC model in six dimensions. Calculation of the subcritical indices $\chi(w) = \Sigma_{k=0}^{\infty} \chi_k (1 - w/w_c)^{-\gamma_k}$; $w_c = 0{\cdot}920979$.

Number of terms	γ_0	χ_0	γ_1	χ_1	γ_2	χ_2
8	1·0126	1·129	0·332	−0·127	−2·33	$-1{\cdot}7\times10^{-3}$
10	1·0105	1·1386	0·367	−0·136	−2·05	$-2{\cdot}2\times10^{-3}$
12	1·0105	1·1386	0·367	−0·136	−2·05	$-2{\cdot}2\times10^{-3}$

4. Discussion and conclusion

One of the most interesting aspects of the method of subdominant indices is that, by spreading out the possible coalescing singularities of the magnetic susceptibilities at $T = T_c$, it allows one to calculate the critical index itself free of interferences coming from these singularities.

Therefore a more reliable result is expected. Furthermore, and even more important, is the fact that the stability of the poles of the various Padé approximants associated with the inverse Laplace transform of $\chi(w)$ in the variable $\log(1 - w/w_c)$ give a rather clear insight into the analytical structure of $\chi(w)$ near w_c. In two dimensions, using the exact critical temperature and the 22 terms of the expansion of the square lattice, we have obtained

$$\begin{aligned} \gamma &= 1{\cdot}74995 \qquad &&(\text{exact } 1{\cdot}75000), \\ \gamma_s &= 0{\cdot}7567 \qquad &&(\text{exact } 0{\cdot}7500) \end{aligned} \tag{4.1}$$

for the leading critical index and the first subdominant one.

In three dimensions we have obtained

$$\gamma = 1{\cdot}2506 \pm 0{\cdot}0015, \qquad \gamma_s = 0{\cdot}42 \pm 0{\cdot}11 \tag{4.2}$$

using the 22 terms of the expansion of the diamond lattice. All the other models known with less terms give values consistent with the previous ones with larger uncertainties (with the exception of the FCC model, which gives rise to slightly lower values for both γ and γ_s).

The value $\gamma = 1{\cdot}2506 \pm 0{\cdot}0015$ remains higher than the field theoretical calculations, which give (Baker *et al* 1976, Le Guillou and Zinn-Justin 1977) $\gamma = 1{\cdot}241 \pm 0{\cdot}002$.

However, one remarks in our method that, when γ is computed with a small number of terms of the high-temperature series, its value is significantly smaller than $1{\cdot}250$. In fact, the value of γ computed for the diamond with 8 and 10 terms seems to be stable and gives

$$\gamma^{(8)} = 1{\cdot}2406, \qquad \gamma^{(10)} = 1{\cdot}2405. \tag{4.3}$$

If we had known only 10 terms, we would have concluded a good agreement with the field theoretical approach.

Our conclusion is that this persistent discrepancy remains to be explained. Gaunt and Sykes (1979) recently made a similar statement.

We come now, to the discussion of the first subdominant index.

The field theoretical approach (Wegner 1972, Brézin *et al* 197,6) predicts the existence of a subdominant universal singularity with the value

$$\gamma_s = \gamma - \omega\nu = 1{\cdot}24 - 0{\cdot}49 = 0{\cdot}75.$$

Another possibility is the existence of a subdominant 'regular' singularity at $\gamma - 1 \simeq 0{\cdot}25$. Our results are more in agreement with the last possibility; $\gamma_s = 0{\cdot}25$. However, the amplitude remains very weak (of the order of 5% of the leading one in the diamond case). Therefore the uncertainties in the value of the subdominant index remain large. We have also performed additional tests by superimposing a singularity at $\gamma - 1$ with an amplitude of 10% of the dominant one and obtained a very neat stabilisation around $0{\cdot}25 \pm 0{\cdot}10$. Therefore we believe that the subdominant singularity has a small amplitude (less than 5%) and a position closer to $0{\cdot}25$ than to the field theoretical predicted value. In the two-dimensional case the subdominant singularity has an amplitude of 45% of the dominant one, which allows a much better precision in the determination of the corresponding index. We want to point out that our results rely upon the choice of subdiagonal Padé approximants. In addition we have used the values of critical temperature given in Domb (1974) and McKenzie (1975). Further analysis will be required if the critical temperature presently admitted happens to be modified after calculation of a larger number of terms in the series expansion. The relation between the dominant index γ and the critical temperature is clearly exhibited in a recent analysis of the BCC lattice (Fischer and Au-Yang 1979). In the FCC lattice it has also been observed recently (McKenzie 1979) that, when the critical parameter w_c is decreased, γ decreases and γ_s increases. Our results show variations in the same direction.

A possible explanation for the small discrepancy between field theory and the high-temperature analysis calculation of γ, as well as for the larger discrepancy in γ_s, is that, as suggested by Baker (1977), hyperscaling may be slightly violated for the spin-$\frac{1}{2}$ Ising models. Such a violation, even if very small, is clearly greatly enhanced if one considers sensitive parameters such as the subdominant indices.

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